Areas and Paper Folding

Dr Stephen Arnold

Level

Year 7/8, Years 8 and 9, Years 10 to 12 (Geometry (extension) and Calculus (extension))

Time required

Between 45 and 90 minutes, depending on the number of modules chosen.

Activity overview

This activity seeks the triangle of largest area produced by folding a sheet of paper, initially through direct measurement and calculation, followed by data collection leading to a graphical model. Beyond Pre-Algebra, students use Pythagoras' Theorem to build a non-linear algebraic model, which may be verified against both the graphical model provided by the class data, and an interactive geometric model, linked to a dynamic graph. Extension activities include seeking to explain their results using both calculus methods and a geometric consideration of the relationship between the angle of the fold and the area of the triangle.

Background

This simple paper folding activity offers a powerful learning experience which builds across the school years and links physical, concrete, geometric, graphical and algebraic representations within a context of measurement, calculation, sampling and data collection, algebraic modeling and interpretation of mathematical results.

Concepts

Measurement of length, areas of rectangles and triangles, data collection and representation, Pythagoras' Theorem, expanding and simplifying of algebraic expressions leading to graphs of non-linear functions, interpretation of graphs and algebraic models.

Teacher preparation

This investigation offers an ongoing extendable activity which may be introduced in the early years of high school, and then revisited at greater depth in subsequent years.

- *At Year 7/8 level, it serves to consolidate earlier work on measurement and area, and offers a suitable introduction to data collection and interpretation of graphs.*
- *For Years 8 to 10 students, it offers a suitable consolidation and extension of Pythagoras' Theorem, providing a context for multiple practice of measuring sides of triangles and verifying other sides using Pythagoras.*
- *Years 10 to 12 students will review and consolidate earlier work (as described above) and attempt to build an algebraic model using Pythagoras and basic algebraic manipulations. This model may be verified against both the data provided by class measurements and that generated by the geometric model provided by TI-Nspire CAS.*
- *Extensions to Geometry and Calculus will serve to review and consolidate earlier work, and provide a strong context for work on Right-Angled Trigonometry and differential calculus for optimization.*

TI-Nspire applications

Calculator, Lists & Spreadsheets, Notes, Graphs & Geometry

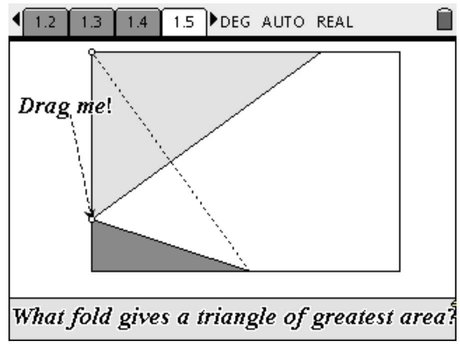
Stephen Arnold

Grade level: secondary (Years 7-12)
Subject: mathematics
Time required: 45 minutes

Areas and Paper Folding

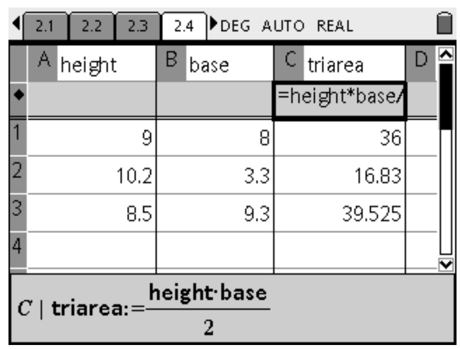
Take a sheet of paper in landscape mode and fold the top left corner to a point along the base, forming a small triangle in the bottom left corner, as shown.

At what point along the side should I make my fold to create the triangle with the largest area?



Step 1: Carefully measure the length and width of a sheet of paper and calculate its area. Check that a partner agrees with you!

Step 2: Fold the page as shown. Measure the triangle you make and use the base and height to calculate the area. Check that a partner agrees with you!

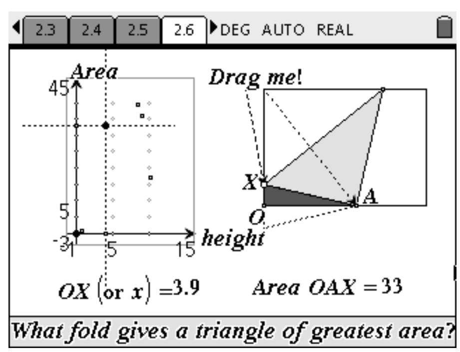


	A height	B base	C triarea	D
◆			=height*base/	
1	9	8	36	
2	10.2	3.3	16.83	
3	8.5	9.3	39.525	
4				

$C \mid \text{triarea} := \frac{\text{height} \cdot \text{base}}{2}$

Step 3: Enter the height and base data for the entire class into a *Lists & Spreadsheet* page. Label the columns "height" and "base" and then define column C (area) using the formula "height*base/2". Check the area you calculated!

Step 4: Use the *TI-Nspire* geometric model to compare the class data with the data generated by dragging the point **X** on the model.



Exercises: Pre-Algebra (Year 8)

1. As carefully as you can, measure the length and width of the page in front of you. How many one inch squares would be needed to cover the page? Now fold as shown and measure the height and base of the triangle formed – find its area.

2. Carefully explain how the graph and the geometric model help us to answer our question.

3. Using your class data and the geometric model, which dimensions appear to produce the largest triangle?

4. How would you account for the differences between some of our class data and the data generated by the geometric model?

Sample Answers

1. *You should find the page is 21 cm wide and 29.7 cm long. It would take over 620 one cm squares to cover the page.*

2. *The geometric model shows the possible different folds which can be made, and is linked to the graph, which shows the relationship between the height (OX) and the area of the triangle OAX. It clearly shows that there is a maximum area. As the height changes the area increases and then decreases.*

3. *The greatest area seems to occur when the height of the triangle is around 7 cm. This is supported by both the class data and the geometric model.*

4. *The most likely explanation would involve errors in our class measurements. These may have resulted from actual mistakes, or just from rounding errors when we read off the lengths of height and base. Less accurate measurements are clearly seen on the graph, as those points which fall to either side of the curved path.*

Exercises: Years 9 and 10

1. For what values of OX does the triangle OAX cease to exist? Why?

2. Find an algebraic expression for the relationship between the **height** (x) and the **hypotenuse**, h(x), of our triangle.

3. Use the tools provided to explore the relationship between the **height** and the **base** of our triangle.

4. Can you describe how we might use Pythagoras' Theorem to find the length of the **base** of our triangle?

Sample Answers

1. *The triangle vanishes when the height is 0, and for any height more than halfway up the page (10.5 cm). The 0 result is obvious; the halfway height occurs since if a fold is made more than halfway up the side, then the top corner will not reach the opposite (base) side.*

2. *The hypotenuse of our triangle, AX, is exactly the same as the remainder of the short side of the page once* x *is removed, since it is simply this remainder folded down (See the geometric model). The width of the page is 21 cm, and so the length of the side after removing x* (and hence the length of AX) is simply 21 – x.

3. *This relationship is not linear, but connected using Pythagoras' Theorem.*

4. *Our triangle OAX is right-angled at O, with height* x *and hypotenuse 21 –* x. *Since, by Pythagoras,*

$$\text{Hypotenuse}^2 = \text{Height}^2 + \text{Base}^2$$

Then Base = $\sqrt{Hypotenuse^2 - Height^2}$

OR OA = $\sqrt{AX^2 - x^2}$

Exercises: Years 11 - 13

1. Using Pythagoras' Theorem, build an algebraic model for the area of our triangle in terms of the height *x*.
2. If CAS is available, use substituting, expanding and simplifying to give the formula for the area in terms of "*x*".
3. Carefully explain how this algebraic model helps to answer our question.

Sample Answers

The base is given by OA = $\sqrt{AX^2 - x^2}$ *and the height is* x*. Hence,*

$$Area(x) = \frac{1}{2} * x * \sqrt{AX^2 - x^2}$$

But AX = 21 – x, and so:

$$Area(x) = \frac{1}{2} * x * \sqrt{(21 - x)^2 - x^2}$$

Now $(21 - x)^2 = x^2 - 42x + 441$

And so $(21 - x)^2 - x^2 = 441 - 42x$*, giving*

$$Area(x) = \frac{1}{2} * x * \sqrt{441 - 42x}$$

Having an algebraic model allows us to evaluate the area for any value of x*, and so to perform very accurate calculations, without the problems of measurement errors or rounding approximations.*

We may even use this model to calculate our value of x *for which the area is greatest, since this must occur at the turning point of the curve (when the derivative of the area equals zero, i.e. at x = 7). Every other value of x will result either in an area that is undefined, or smaller than that when x = 7.This may be easily seen graphically and our algebraic model may be verified against the class data and the geometric model by graphing.*

Exercises: Geometry (CAS Extension)

Sample Answers

1. Measure the angle OXA. Describe how we could use this angle and the height to find the length of the base of triangle OAX.

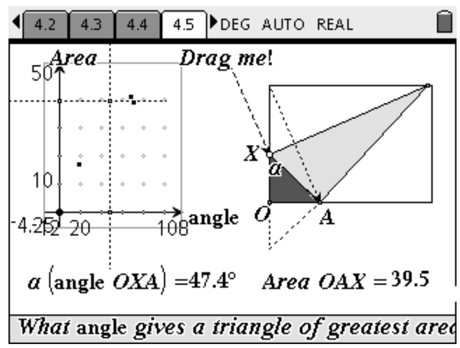
Let angle OXA = .

Then tan( ) = base/height, and so

base = height x tan( )

OA = x tan( ).

2. Explore the relationship between the angle and the area of the triangle, first by measurement and data collection, and then using the geometric model provided.



3. Clearly explain how these models may help us to answer our question.

As the angle varies between 0 and 90 degrees, the area increases and then decreases, with a clear maximum value.

4. Find a formula for the area of the triangle OAX in terms of the angle and the side, *x*.

Area( ) = $\frac{1}{2} * x * (x * \tan(\alpha))$

Area( ) = $\frac{1}{2} x^2 \tan(\alpha))$

5. What value of appears to give the triangle of greatest area? Why?

The maximum area appears to occur around = 60 degrees. At this value, the triangle OAX forms one half of an equilateral triangle, which is the optimum shape, greatest area for least perimeter.

Exercises: Calculus (CAS Extension)

1. Use calculus to find the **height** of the fold (x) which results in the triangle of greatest area (if CAS is not available then the numerical function maximum (nFmax) command will also work).

2. In the same way, find the **angle** of the fold which gives the largest triangle.

Sample Answers

Define AX(x) = 8.5 – x

Define OA(x) = $\sqrt{AX(x)^2 - x^2}$

Define Area(x) = $\frac{1}{2} * x * OA(x)$

Solve $(\frac{d}{dx}(Area(x)) = 0, x)$ *=> x = 7*

Define base (OA) = x tan( )

Define hypotenuse = $\frac{x}{\cos(\alpha)}$ *= 21 – x*

$$x = \frac{21\cos(\alpha)}{\cos(\alpha)+1}$$

Define OA( ) = $\frac{21\cos(\alpha)\tan(\alpha)}{\cos(\alpha)+1}$

Define Area2( ) = $\frac{1}{2} * x * OA(\alpha)$

Area2( ) = $\frac{1}{2}\frac{21^2\cos(\alpha)\sin(\alpha)}{(\cos(\alpha)+1)^2}$

Solve($(\frac{d}{d\alpha}(Area2(\alpha)) = 0, \alpha) \mid \alpha > 0 and \alpha < 90$

⇨ *= 60 degrees.*

3. Can you explain why the largest triangle results from this particular fold (You might try to find the general solution, for a page of width "*w*").

   How might the angle help?

Define OA(x) = w-x

$$Solve(\frac{d}{dx}Area(x) = 0, x) \Rightarrow x = \frac{w}{3} or x = 0$$

In this way we confirm that the optimal solution involves a fold exactly one third of the width of the page from the bottom corner. While this is an elegant solution, it does not explain WHY it is the best.

For this, we need to refer to the geometric solution, involving the angle of 60 degrees, which dictates that the largest triangle will be the one that is exactly half of an equilateral triangle. The equilateral triangle is the most efficient of all triangles in terms of giving the greatest area for the smallest perimeter, and this helps to explain why it is the best shape for our problem.

4. Give a brief but clear summary of your conclusions.

Any fold between the bottom left corner and halfway up the side will produce a triangle, but the triangle with greatest area results from a fold exactly one third of the way up the side of the page. This is because this fold forms an angle of 60 degrees with the side of the page, and, as such, the triangle forms half of an equilateral triangle, which has maximum area for minimum perimeter.

Assessment and evaluation

- *The worksheets provided above for each level offer both a teaching and assessment sequence which is clearly differentiated for learners at different levels.*

Activity extensions

- *Extensions are provided within the teaching and assessment sequence provided.*